

CHƯƠNG 1: CÔNG THỨC LƯỢNG GIÁC

I. Định nghĩa

Trên mặt phẳng Oxy cho đường tròn lượng giác tâm O bán kính R=1 và điểm M trên đường tròn lượng giác mà số $\widehat{AM} = \beta$ với $0 \leq \beta \leq 2\pi$

Đặt $\alpha = \beta + k2\pi, k \in \mathbb{Z}$

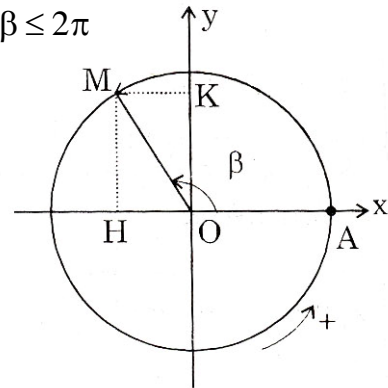
Ta định nghĩa:

$$\sin \alpha = \overline{OK}$$

$$\cos \alpha = \overline{OH}$$

$$\operatorname{tg} \alpha = \frac{\sin \alpha}{\cos \alpha} \text{ với } \cos \alpha \neq 0$$

$$\operatorname{cot} g \alpha = \frac{\cos \alpha}{\sin \alpha} \text{ với } \sin \alpha \neq 0$$



II. Bảng giá trị lượng giác của một số cung (hay góc) đặc biệt

Góc α	$0(0^\circ)$	$\frac{\pi}{6}(30^\circ)$	$\frac{\pi}{4}(45^\circ)$	$\frac{\pi}{3}(60^\circ)$	$\frac{\pi}{2}(90^\circ)$
Giá trị					
$\sin \alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\operatorname{tg} \alpha$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	\parallel
$\operatorname{cot} g \alpha$	\parallel	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

III. Hệ thức cơ bản

$$\sin^2 \alpha + \cos^2 \alpha = 1$$

$$1 + \operatorname{tg}^2 \alpha = \frac{1}{\cos^2 \alpha} \text{ với } \alpha \neq \frac{\pi}{2} + k\pi (k \in \mathbb{Z})$$

$$1 + \operatorname{cot} g^2 \alpha = \frac{1}{\sin^2 \alpha} \text{ với } \alpha \neq k\pi (k \in \mathbb{Z})$$

IV. Cung liên kết (Cách nhớ: cos đối, sin bù, tang sai π ; phụ chéo)

a. Đối nhau: α và $-\alpha$

$$\sin(-\alpha) = -\sin \alpha$$

$$\cos(-\alpha) = \cos \alpha$$

$$\operatorname{tg}(-\alpha) = -\operatorname{tg}(\alpha)$$

$$\operatorname{cot} g(-\alpha) = -\operatorname{cot} g(\alpha)$$

b. Bù nhau: α và $\pi - \alpha$

$$\begin{aligned}\sin(\pi - \alpha) &= \sin \alpha \\ \cos(\pi - \alpha) &= -\cos \alpha \\ \operatorname{tg}(\pi - \alpha) &= -\operatorname{tg} \alpha \\ \operatorname{cot} g(\pi - \alpha) &= -\operatorname{cot} g \alpha\end{aligned}$$

c. Sai nhau π : α và $\pi + \alpha$

$$\begin{aligned}\sin(\pi + \alpha) &= -\sin \alpha \\ \cos(\pi + \alpha) &= -\cos \alpha \\ \operatorname{tg}(\pi + \alpha) &= \operatorname{tg} \alpha \\ \operatorname{cot} g(\pi + \alpha) &= \operatorname{cot} g \alpha\end{aligned}$$

d. Phụ nhau: α và $\frac{\pi}{2} - \alpha$

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= \cos \alpha \\ \cos\left(\frac{\pi}{2} - \alpha\right) &= \sin \alpha \\ \operatorname{tg}\left(\frac{\pi}{2} - \alpha\right) &= \operatorname{cot} g \alpha \\ \operatorname{cot} g\left(\frac{\pi}{2} - \alpha\right) &= \operatorname{tg} \alpha\end{aligned}$$

e. Sai nhau $\frac{\pi}{2}$: α và $\frac{\pi}{2} + \alpha$

$$\begin{aligned}\sin\left(\frac{\pi}{2} + \alpha\right) &= \cos \alpha \\ \cos\left(\frac{\pi}{2} + \alpha\right) &= -\sin \alpha \\ \operatorname{tg}\left(\frac{\pi}{2} + \alpha\right) &= -\operatorname{cot} g \alpha \\ \operatorname{cot} g\left(\frac{\pi}{2} + \alpha\right) &= -\operatorname{tg} \alpha\end{aligned}$$

f.

$$\sin(x + k\pi) = (-1)^k \sin x, k \in \mathbb{Z}$$

$$\cos(x + k\pi) = (-1)^k \cos x, k \in \mathbb{Z}$$

$$\operatorname{tg}(x + k\pi) = \operatorname{tg}x, k \in \mathbb{Z}$$

$$\operatorname{cot}g(x + k\pi) = \operatorname{cot}g x$$

V. Công thức cộng

$$\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\operatorname{tg}(a \pm b) = \frac{\operatorname{tga} \pm \operatorname{tgb}}{1 \mp \operatorname{tgatgb}}$$

VI. Công thức nhân đôi

$$\sin 2a = 2 \sin a \cos a$$

$$\cos 2a = \cos^2 a - \sin^2 a = 1 - 2 \sin^2 a = 2 \cos^2 a - 1$$

$$\operatorname{tg}2a = \frac{2 \operatorname{tga}}{1 - \operatorname{tg}^2 a}$$

$$\operatorname{cot}g2a = \frac{\operatorname{cot}g^2 a - 1}{2 \operatorname{cot}g a}$$

VII. Công thức nhân ba:

$$\sin 3a = 3 \sin a - 4 \sin^3 a$$

$$\cos 3a = 4 \cos^3 a - 3 \cos a$$

VIII. Công thức hạ bậc:

$$\sin^2 a = \frac{1}{2}(1 - \cos 2a)$$

$$\cos^2 a = \frac{1}{2}(1 + \cos 2a)$$

$$\operatorname{tg}^2 a = \frac{1 - \cos 2a}{1 + \cos 2a}$$

IX. Công thức chia đôi

$$\text{Đặt } t = \operatorname{tg} \frac{a}{2} \text{ (với } a \neq \pi + k2\pi)$$

$$\sin a = \frac{2t}{1+t^2}$$

$$\cos a = \frac{1-t^2}{1+t^2}$$

$$\operatorname{tga} = \frac{2t}{1-t^2}$$

X. Công thức biến đổi tổng thành tích

$$\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}$$

$$\cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}$$

$$\sin a + \sin b = 2 \cos \frac{a+b}{2} \sin \frac{a-b}{2}$$

$$\sin a - \sin b = 2 \cos \frac{a+b}{2} \sin \frac{a-b}{2}$$

$$\operatorname{tga} \pm \operatorname{tgb} = \frac{\sin(a \pm b)}{\cos a \cos b}$$

$$\operatorname{cotga} \pm \operatorname{cotgb} = \frac{\sin(b \pm a)}{\sin a \cdot \sin b}$$

XI. Công thức biến đổi tích thành tổng

$$\cos a \cdot \cos b = \frac{1}{2} [\cos(a+b) + \cos(a-b)]$$

$$\sin a \cdot \sin b = \frac{-1}{2} [\cos(a+b) - \cos(a-b)]$$

$$\sin a \cdot \cos b = \frac{1}{2} [\sin(a+b) + \sin(a-b)]$$

Bài 1: Chứng minh $\frac{\sin^4 a + \cos^4 a - 1}{\sin^6 a + \cos^6 a - 1} = \frac{2}{3}$

Ta có:

$$\sin^4 a + \cos^4 a - 1 = (\sin^2 a + \cos^2 a)^2 - 2\sin^2 a \cos^2 a - 1 = -2\sin^2 a \cos^2 a$$

Và:

$$\begin{aligned} \sin^6 a + \cos^6 a - 1 &= (\sin^2 a + \cos^2 a)(\sin^4 a - \sin^2 a \cos^2 a + \cos^4 a) - 1 \\ &= \sin^4 a + \cos^4 a - \sin^2 a \cos^2 a - 1 \\ &= (1 - 2\sin^2 a \cos^2 a) - \sin^2 a \cos^2 a - 1 \\ &= -3\sin^2 a \cos^2 a \end{aligned}$$

$$\text{Do đó: } \frac{\sin^4 a + \cos^4 a - 1}{\sin^6 a + \cos^6 a - 1} = \frac{-2\sin^2 a \cos^2 a}{-3\sin^2 a \cos^2 a} = \frac{2}{3}$$

Bài 2: Rút gọn biểu thức $A = \frac{1 + \cos x}{\sin x} = \left[1 + \frac{(1 - \cos x)^2}{\sin^2 x} \right]$

Tính giá trị A nếu $\cos x = -\frac{1}{2}$ và $\frac{\pi}{2} < x < \pi$

$$\text{Ta có: } A = \frac{1 + \cos x}{\sin x} \left(\frac{\sin^2 x + 1 - 2\cos x + \cos^2 x}{\sin^2 x} \right)$$

$$\Leftrightarrow A = \frac{1 + \cos x}{\sin x} \cdot \frac{2(1 - \cos x)}{\sin^2 x}$$

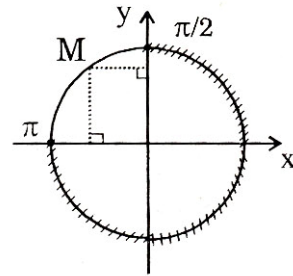
$$\Leftrightarrow A = \frac{2(1 - \cos^2 x)}{\sin^3 x} = \frac{2\sin^2 x}{\sin^3 x} = \frac{2}{\sin x} \quad (\text{với } \sin x \neq 0)$$

$$\text{Ta có: } \sin^2 x = 1 - \cos^2 x = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\text{Do: } \frac{\pi}{2} < x < \pi \text{ nên } \sin x > 0$$

$$\text{Vậy } \sin x = \frac{\sqrt{3}}{2}$$

$$\text{Do đó } A = \frac{2}{\sin x} = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$$



Bài 3: Chứng minh các biểu thức sau đây không phụ thuộc x:

a. $A = 2\cos^4 x - \sin^4 x + \sin^2 x \cos^2 x + 3\sin^2 x$

b. $B = \frac{2}{\text{tg}x - 1} + \frac{\text{cot}gx + 1}{\text{cot}gx - 1}$

a. Ta có:

$$A = 2\cos^4 x - \sin^4 x + \sin^2 x \cos^2 x + 3\sin^2 x$$

$$\Leftrightarrow A = 2\cos^4 x - (1 - \cos^2 x)^2 + (1 - \cos^2 x)\cos^2 x + 3(1 - \cos^2 x)$$

$$\Leftrightarrow A = 2\cos^4 x - (1 - 2\cos^2 x + \cos^4 x) + \cos^2 x - \cos^4 x + 3 - 3\cos^2 x$$

$$\Leftrightarrow A = 2 \quad (\text{không phụ thuộc } x)$$

b. Với điều kiện $\sin x \cdot \cos x \neq 0, \text{tg}x \neq 1$

$$\text{Ta có: } B = \frac{2}{\text{tg}x - 1} + \frac{\text{cot}gx + 1}{\text{cot}gx - 1}$$

$$\Leftrightarrow B = \frac{2}{\operatorname{tg}x - 1} + \frac{\frac{1}{\operatorname{tg}x} + 1}{\frac{1}{\operatorname{tg}x} - 1} = \frac{2}{\operatorname{tg}x - 1} + \frac{1 + \operatorname{tg}x}{1 - \operatorname{tg}x}$$

$$\Leftrightarrow B = \frac{2 - (1 - \operatorname{tg}x)}{\operatorname{tg}x - 1} = \frac{1 - \operatorname{tg}x}{\operatorname{tg}x - 1} = -1 \text{ (không phụ thuộc vào } x)$$

Bài 4: Chứng minh

$$\frac{1 + \cos a}{2 \sin a} \left[1 - \frac{(1 - \cos a)^2}{\sin^2 a} \right] + \frac{\cos^2 b - \sin^2 c}{\sin^2 b \sin^2 c} - \cot g^2 b \cot g^2 c = \cot ga - 1$$

Ta có:

$$\begin{aligned} & * \frac{\cos^2 b - \sin^2 c}{\sin^2 b \cdot \sin^2 c} - \cot g^2 b \cdot \cot g^2 c \\ &= \frac{\cot g^2 b}{\sin^2 c} - \frac{1}{\sin^2 b} - \cot g^2 b \cot g^2 c \\ &= \cot g^2 b (1 + \cot g^2 c) - (1 + \cot g^2 b) - \cot g^2 b \cot g^2 c = -1 \quad (1) \end{aligned}$$

$$\begin{aligned} & * \frac{1 + \cos a}{2 \sin a} \left[1 - \frac{(1 - \cos a)^2}{\sin^2 a} \right] \\ &= \frac{1 + \cos a}{2 \sin a} \left[1 - \frac{(1 - \cos a)^2}{1 - \cos^2 a} \right] \\ &= \frac{1 + \cos a}{2 \sin a} \left[1 - \frac{1 - \cos a}{1 + \cos a} \right] \\ &= \frac{1 + \cos a}{2 \sin a} \cdot \frac{2 \cos a}{1 + \cos a} = \cot ga \quad (2) \end{aligned}$$

Lấy (1) + (2) ta được điều phải chứng minh xong.

Bài 5: Cho ΔABC tùy ý với ba góc đều là nhọn.
Tìm giá trị nhỏ nhất của $P = \operatorname{tg}A \cdot \operatorname{tg}B \cdot \operatorname{tg}C$

Ta có: $A + B = \pi - C$

Nên: $\operatorname{tg}(A + B) = -\operatorname{tg}C$

$$\Leftrightarrow \frac{\operatorname{tg}A + \operatorname{tg}B}{1 - \operatorname{tg}A \cdot \operatorname{tg}B} = -\operatorname{tg}C$$

$$\Leftrightarrow \operatorname{tg}A + \operatorname{tg}B = -\operatorname{tg}C + \operatorname{tg}A \cdot \operatorname{tg}B \cdot \operatorname{tg}C$$

Vậy: $P = \operatorname{tg}A \cdot \operatorname{tg}B \cdot \operatorname{tg}C = \operatorname{tg}A + \operatorname{tg}B + \operatorname{tg}C$

Áp dụng bất đẳng thức Cauchy cho ba số dương $\operatorname{tg}A, \operatorname{tg}B, \operatorname{tg}C$ ta được

$$\operatorname{tg}A + \operatorname{tg}B + \operatorname{tg}C \geq 3\sqrt[3]{\operatorname{tg}A \cdot \operatorname{tg}B \cdot \operatorname{tg}C}$$

$$\Leftrightarrow P \geq 3\sqrt[3]{P}$$

$$\Leftrightarrow \sqrt[3]{P^2} \geq 3$$

$$\Leftrightarrow P \geq 3\sqrt{3}$$

$$\text{Dấu "=" xảy ra} \Leftrightarrow \begin{cases} \text{tg}A = \text{tg}B = \text{tg}C \\ 0 < A, B, C < \frac{\pi}{2} \end{cases} \Leftrightarrow A = B = C = \frac{\pi}{3}$$

$$\text{Do đó: } \text{Min}P = 3\sqrt{3} \Leftrightarrow A = B = C = \frac{\pi}{3}$$

Bài 6: Tìm giá trị lớn nhất và nhỏ nhất của

$$a/ y = 2 \sin^8 x + \cos^4 2x$$

$$b/ y = \sqrt[4]{\sin x} - \sqrt{\cos x}$$

$$a/ \text{Ta có: } y = 2 \left(\frac{1 - \cos 2x}{2} \right)^4 + \cos^4 2x$$

Đặt $t = \cos 2x$ với $-1 \leq t \leq 1$ thì

$$y = \frac{1}{8}(1-t)^4 + t^4$$

$$\Rightarrow y' = -\frac{1}{2}(1-t)^3 + 4t^3$$

$$\text{Ta có: } y' = 0 \Leftrightarrow (1-t)^3 = 8t^3$$

$$\Leftrightarrow 1-t = 2t$$

$$\Leftrightarrow t = \frac{1}{3}$$

$$\text{Ta có } y(1) = 1; y(-1) = 3; y\left(\frac{1}{3}\right) = \frac{1}{27}$$

$$\text{Do đó: } \text{Max}_{x \in \mathbb{R}} y = 3 \text{ và } \text{Min}_{x \in \mathbb{R}} y = \frac{1}{27}$$

b/ Do điều kiện: $\sin x \geq 0$ và $\cos x \geq 0$ nên miền xác định

$$D = \left[k2\pi, \frac{\pi}{2} + k2\pi \right] \text{ với } k \in \mathbb{Z}$$

Đặt $t = \sqrt{\cos x}$ với $0 \leq t \leq 1$ thì $t^4 = \cos^2 x = 1 - \sin^2 x$

$$\text{Nên } \sin x = \sqrt{1-t^4}$$

$$\text{Vậy } y = \sqrt[8]{1-t^4} - t \text{ trên } D' = [0, 1]$$

$$\text{Thì } y' = \frac{-t^3}{2 \cdot \sqrt[8]{(1-t^4)^7}} - 1 < 0 \quad \forall t \in [0, 1]$$

Nên y giảm trên $[0, 1]$. Vậy: $\max_{x \in D} y = y(0) = 1$, $\min_{x \in D} y = y(1) = -1$

Bài 7: Cho hàm số $y = \sqrt{\sin^4 x + \cos^4 x - 2m \sin x \cos x}$

Tìm giá trị m để y xác định với mọi x

Xét $f(x) = \sin^4 x + \cos^4 x - 2m \sin x \cos x$

$$f(x) = (\sin^2 x + \cos^2 x)^2 - m \sin 2x - 2 \sin^2 x \cos^2 x$$

$$f(x) = 1 - \frac{1}{2} \sin^2 2x - m \sin 2x$$

Đặt : $t = \sin 2x$ với $t \in [-1, 1]$

y xác định $\forall x \Leftrightarrow f(x) \geq 0 \forall x \in \mathbb{R}$

$$\Leftrightarrow 1 - \frac{1}{2} t^2 - mt \geq 0 \quad \forall t \in [-1, 1]$$

$$\Leftrightarrow g(t) = t^2 + 2mt - 2 \leq 0 \quad \forall t \in [-1, 1]$$

Do $\Delta' = m^2 + 2 > 0 \quad \forall m$ nên $g(t)$ có 2 nghiệm phân biệt t_1, t_2

Lúc đó

	t	t_1	t_2
$g(t)$	+	0	-

Do đó : yêu cầu bài toán $\Leftrightarrow t_1 \leq -1 < 1 \leq t_2$

$$\Leftrightarrow \begin{cases} 1g(-1) \leq 0 \\ 1g(1) \leq 0 \end{cases} \Leftrightarrow \begin{cases} -2m - 1 \leq 0 \\ 2m - 1 \leq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} m \geq \frac{-1}{2} \\ m \leq \frac{1}{2} \end{cases} \Leftrightarrow -\frac{1}{2} \leq m \leq \frac{1}{2}$$

Cách khác :

$$g(t) = t^2 + 2mt - 2 \leq 0 \quad \forall t \in [-1, 1]$$

$$\Leftrightarrow \max_{t \in [-1, 1]} g(t) \leq 0 \Leftrightarrow \max \{g(-1), g(1)\} \leq 0$$

$$\Leftrightarrow \max \{-2m - 1, -2m + 1\} \leq 0 \Leftrightarrow \begin{cases} m \geq \frac{-1}{2} \\ m \leq \frac{1}{2} \end{cases}$$

$$\Leftrightarrow -\frac{1}{2} \leq m \leq \frac{1}{2}$$

Bài 8 : Chứng minh $A = \sin^4 \frac{\pi}{16} + \sin^4 \frac{3\pi}{16} + \sin^4 \frac{5\pi}{16} + \sin^4 \frac{7\pi}{16} = \frac{3}{2}$

Ta có : $\sin \frac{7\pi}{16} = \sin \left(\frac{\pi}{2} - \frac{\pi}{16} \right) = \cos \frac{\pi}{16}$

$$\sin \frac{5\pi}{16} = \cos \left(\frac{\pi}{2} - \frac{5\pi}{16} \right) = \cos \frac{3\pi}{16}$$

Mặt khác : $\sin^4 \alpha + \cos^4 \alpha = (\sin^2 \alpha + \cos^2 \alpha)^2 - 2 \sin^2 \alpha \cos^2 \alpha$
 $= 1 - 2 \sin^2 \alpha \cos^2 \alpha$
 $= 1 - \frac{1}{2} \sin^2 2\alpha$

$$\begin{aligned}
\text{Do đó : } A &= \sin^4 \frac{\pi}{16} + \sin^4 \frac{7\pi}{16} + \sin^4 \frac{3\pi}{16} + \sin^4 \frac{5\pi}{16} \\
&= \left(\sin^4 \frac{\pi}{16} + \cos^4 \frac{\pi}{16} \right) + \left(\sin^4 \frac{3\pi}{16} + \cos^4 \frac{3\pi}{16} \right) \\
&= \left(1 - \frac{1}{2} \sin^2 \frac{\pi}{8} \right) + \left(1 - \frac{1}{2} \sin^2 \frac{3\pi}{8} \right) \\
&= 2 - \frac{1}{2} \left(\sin^2 \frac{\pi}{8} + \sin^2 \frac{3\pi}{8} \right) \\
&= 2 - \frac{1}{2} \left(\sin^2 \frac{\pi}{8} + \cos^2 \frac{\pi}{8} \right) \quad \left(\text{do } \sin \frac{3\pi}{8} = \cos \frac{\pi}{8} \right) \\
&= 2 - \frac{1}{2} = \frac{3}{2}
\end{aligned}$$

Bài 9 : Chứng minh : $16 \sin 10^\circ \cdot \sin 30^\circ \cdot \sin 50^\circ \cdot \sin 70^\circ = 1$

$$\begin{aligned}
\text{Ta có : } A &= \frac{A \cos 10^\circ}{\cos 10^\circ} = \frac{1}{\cos 10^\circ} (16 \sin 10^\circ \cos 10^\circ) \sin 30^\circ \cdot \sin 50^\circ \cdot \sin 70^\circ \\
\Leftrightarrow A &= \frac{1}{\cos 10^\circ} (8 \sin 20^\circ) \left(\frac{1}{2} \right) \cos 40^\circ \cdot \cos 20^\circ \\
\Leftrightarrow A &= \frac{1}{\cos 10^\circ} (4 \sin 20^\circ \cos 20^\circ) \cdot \cos 40^\circ \\
\Leftrightarrow A &= \frac{1}{\cos 10^\circ} (2 \sin 40^\circ) \cos 40^\circ \\
\Leftrightarrow A &= \frac{1}{\cos 10^\circ} \sin 80^\circ = \frac{\cos 10^\circ}{\cos 10^\circ} = 1
\end{aligned}$$

Bài 10 : Cho $\triangle ABC$. Chứng minh : $\text{tg} \frac{A}{2} \text{tg} \frac{B}{2} + \text{tg} \frac{B}{2} \text{tg} \frac{C}{2} + \text{tg} \frac{C}{2} \text{tg} \frac{A}{2} = 1$

$$\begin{aligned}
\text{Ta có : } \frac{A+B}{2} &= \frac{\pi}{2} - \frac{C}{2} \\
\text{Vậy : } \text{tg} \frac{A+B}{2} &= \text{cotg} \frac{C}{2} \\
\Leftrightarrow \frac{\text{tg} \frac{A}{2} + \text{tg} \frac{B}{2}}{1 - \text{tg} \frac{A}{2} \cdot \text{tg} \frac{B}{2}} &= \frac{1}{\text{tg} \frac{C}{2}} \\
\Leftrightarrow \left[\text{tg} \frac{A}{2} + \text{tg} \frac{B}{2} \right] \text{tg} \frac{C}{2} &= 1 - \text{tg} \frac{A}{2} \text{tg} \frac{B}{2} \\
\Leftrightarrow \text{tg} \frac{A}{2} \text{tg} \frac{C}{2} + \text{tg} \frac{B}{2} \text{tg} \frac{C}{2} + \text{tg} \frac{A}{2} \text{tg} \frac{B}{2} &= 1
\end{aligned}$$

Bài 11 : Chứng minh : $8 + 4 \text{tg} \frac{\pi}{8} + 2 \text{tg} \frac{\pi}{16} + \text{tg} \frac{\pi}{32} = \text{cotg} \frac{\pi}{32} (*)$

$$\text{Ta có : (*)} \Leftrightarrow 8 = \cot g \frac{\pi}{32} - \text{tg} \frac{\pi}{32} - 2\text{tg} \frac{\pi}{16} - 4\text{tg} \frac{\pi}{8}$$

$$\begin{aligned} \text{Mà : } \cot g a - \text{tga} &= \frac{\cos a}{\sin a} - \frac{\sin a}{\cos a} = \frac{\cos^2 a - \sin^2 a}{\sin a \cos a} \\ &= \frac{\cos 2a}{\frac{1}{2} \sin 2a} = 2 \cot g 2a \end{aligned}$$

Do đó :

$$(*) \Leftrightarrow \left[\cot g \frac{\pi}{32} - \text{tg} \frac{\pi}{32} \right] - 2\text{tg} \frac{\pi}{16} - 4\text{tg} \frac{\pi}{8} = 8$$

$$\Leftrightarrow \left[2 \cot g \frac{\pi}{16} - 2\text{tg} \frac{\pi}{16} \right] - 4\text{tg} \frac{\pi}{8} = 8$$

$$\Leftrightarrow 4 \cot g \frac{\pi}{8} - 4\text{tg} \frac{\pi}{8} = 8$$

$$\Leftrightarrow 8 \cot g \frac{\pi}{4} = 8 \text{ (hiển nhiên đúng)}$$

Bài :12 : Chứng minh :

$$\text{a/ } \cos^2 x + \cos^2 \left(\frac{2\pi}{3} + x \right) + \cos^2 \left(\frac{2\pi}{3} - x \right) = \frac{3}{2}$$

$$\text{b/ } \frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \frac{1}{\sin 8x} + \frac{1}{\sin 16x} = \cot gx - \cot g16x$$

$$\begin{aligned} \text{a/ Ta có : } &\cos^2 x + \cos^2 \left(\frac{2\pi}{3} + x \right) + \cos^2 \left(\frac{2\pi}{3} - x \right) \\ &= \frac{1}{2}(1 + \cos 2x) + \frac{1}{2} \left[1 + \cos \left(2x + \frac{4\pi}{3} \right) \right] + \frac{1}{2} \left[1 + \cos \left(\frac{4\pi}{3} - 2x \right) \right] \\ &= \frac{3}{2} + \frac{1}{2} \left[\cos 2x + \cos \left(2x + \frac{4\pi}{3} \right) + \cos \left(\frac{4\pi}{3} - 2x \right) \right] \\ &= \frac{3}{2} + \frac{1}{2} \left[\cos 2x + 2 \cos 2x \cos \frac{4\pi}{3} \right] \\ &= \frac{3}{2} + \frac{1}{2} \left[\cos 2x + 2 \cos 2x \left(-\frac{1}{2} \right) \right] \\ &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} \text{b/ Ta có : } \cot g a - \cot g b &= \frac{\cos a}{\sin a} - \frac{\cos b}{\sin b} = \frac{\sin b \cos a - \sin a \cos b}{\sin a \sin b} \\ &= \frac{\sin(b - a)}{\sin a \sin b} \end{aligned}$$

$$\text{Do đó : } \cot gx - \cot g2x = \frac{\sin(2x - x)}{\sin x \sin 2x} = \frac{1}{\sin 2x} \quad (1)$$

$$\cot g2x - \cot g4x = \frac{\sin(4x - 2x)}{\sin 2x \sin 4x} = \frac{1}{\sin 4x} \quad (2)$$

$$\cot g4x - \cot g8x = \frac{\sin(8x - 4x)}{\sin 4x \sin 8x} = \frac{1}{\sin 8x} \quad (3)$$

$$\cot g8x - \cot g16x = \frac{\sin(16x - 8x)}{\sin 16x \sin 8x} = \frac{1}{\sin 16x} \quad (4)$$

Lấy (1) + (2) + (3) + (4) ta được

$$\cot gx - \cot g16x = \frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \frac{1}{\sin 8x} + \frac{1}{\sin 16x}$$

Bài 13: Chứng minh : $8\sin^3 18^\circ + 8\sin^2 18^\circ = 1$

Ta có: $\sin 18^\circ = \cos 72^\circ$

$$\Leftrightarrow \sin 18^\circ = 2\cos^2 36^\circ - 1$$

$$\Leftrightarrow \sin 18^\circ = 2(1 - 2\sin^2 18^\circ)^2 - 1$$

$$\Leftrightarrow \sin 18^\circ = 2(1 - 4\sin^2 18^\circ + 4\sin^4 18^\circ) - 1$$

$$\Leftrightarrow 8\sin^4 18^\circ - 8\sin^2 18^\circ - \sin 18^\circ + 1 = 0 \quad (1)$$

$$\Leftrightarrow (\sin 18^\circ - 1)(8\sin^3 18^\circ + 8\sin^2 18^\circ - 1) = 0$$

$$\Leftrightarrow 8\sin^3 18^\circ + 8\sin^2 18^\circ - 1 = 0 \quad (\text{do } 0 < \sin 18^\circ < 1)$$

Cách khác :

Chia 2 vế của (1) cho $(\sin 18^\circ - 1)$ ta có

$$(1) \Leftrightarrow 8\sin^2 18^\circ (\sin 18^\circ + 1) - 1 = 0$$

Bài 14: Chứng minh :

$$a/ \sin^4 x + \cos^4 x = \frac{1}{4}(3 + \cos 4x)$$

$$b/ \sin 6x + \cos 6x = \frac{1}{8}(5 + 3\cos 4x)$$

$$c/ \sin^8 x + \cos^8 x = \frac{1}{64}(35 + 28\cos 4x + \cos 8x)$$

$$a/ \text{Ta có: } \sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x$$

$$= 1 - \frac{2}{4}\sin^2 2x$$

$$= 1 - \frac{1}{4}(1 - \cos 4x)$$

$$= \frac{3}{4} + \frac{1}{4}\cos 4x$$

$$b/ \text{Ta có : } \sin 6x + \cos 6x$$

$$= (\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x)$$

$$= (\sin^4 x + \cos^4 x) - \frac{1}{4}\sin^2 2x$$

$$= \left(\frac{3}{4} + \frac{1}{4}\cos 4x\right) - \frac{1}{8}(1 - \cos 4x) \quad (\text{do kết quả câu a})$$

$$= \frac{3}{8}\cos 4x + \frac{5}{8}$$

$$c/ \text{Ta có : } \sin^8 x + \cos^8 x = (\sin^4 x + \cos^4 x)^2 - 2\sin^4 x \cos^4 x$$

$$\begin{aligned}
&= \frac{1}{16} (3 + \cos 4x)^2 - \frac{2}{16} \sin^4 2x \\
&= \frac{1}{16} (9 + 6 \cos 4x + \cos^2 4x) - \frac{1}{8} \left[\frac{1}{2} (1 - \cos 4x) \right]^2 \\
&= \frac{9}{16} + \frac{3}{8} \cos 4x + \frac{1}{32} (1 + \cos 8x) - \frac{1}{32} (1 - 2 \cos 4x + \cos^2 4x) \\
&= \frac{9}{16} + \frac{3}{8} \cos 4x + \frac{1}{32} \cos 8x + \frac{1}{16} \cos 4x - \frac{1}{64} (1 + \cos 8x) \\
&= \frac{35}{64} + \frac{7}{16} \cos 4x + \frac{1}{64} \cos 8x
\end{aligned}$$

Bài 15 : Chứng minh : $\sin 3x \cdot \sin^3 x + \cos 3x \cdot \cos^3 x = \cos^3 2x$

Cách 1:

Ta có : $\sin 3x \cdot \sin^3 x + \cos 3x \cdot \cos^3 x = \cos^3 2x$

$$\begin{aligned}
&= (3 \sin x - 4 \sin^3 x) \sin^3 x + (4 \cos^3 x - 3 \cos x) \cos^3 x \\
&= 3 \sin^4 x - 4 \sin^6 x + 4 \cos^6 x - 3 \cos^4 x \\
&= 3 (\sin^4 x - \cos^4 x) - 4 (\sin^6 x - \cos^6 x) \\
&= 3 (\sin^2 x - \cos^2 x) (\sin^2 x + \cos^2 x) \\
&\quad - 4 (\sin^2 x - \cos^2 x) (\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x) \\
&= -3 \cos 2x + 4 \cos 2x [1 - \sin^2 x \cos^2 x] \\
&= -3 \cos 2x + 4 \cos 2x \left(1 - \frac{1}{4} \sin^2 2x \right) \\
&= \cos 2x \left[-3 + 4 \left(1 - \frac{1}{4} \sin^2 2x \right) \right] \\
&= \cos 2x (1 - \sin^2 2x) \\
&= \cos^3 2x
\end{aligned}$$

Cách 2 :

Ta có : $\sin 3x \cdot \sin^3 x + \cos 3x \cdot \cos^3 x$

$$\begin{aligned}
&= \sin 3x \left(\frac{3 \sin x - \sin 3x}{4} \right) + \cos 3x \left(\frac{3 \cos x + \cos 3x}{4} \right) \\
&= \frac{3}{4} (\sin 3x \sin x + \cos 3x \cos x) + \frac{1}{4} (\cos^2 3x - \sin^2 3x) \\
&= \frac{3}{4} \cos (3x - x) + \frac{1}{4} \cos 6x \\
&= \frac{1}{4} (3 \cos 2x + \cos 3 \cdot 2x) \\
&= \frac{1}{4} (3 \cos 2x + 4 \cos^3 2x - 3 \cos 2x) \text{ (bỏ dòng này cũng được)} \\
&= \cos^3 2x
\end{aligned}$$

Bài 16 : Chứng minh : $\cos 12^\circ + \cos 18^\circ - 4 \cos 15^\circ \cdot \cos 21^\circ \cos 24^\circ = -\frac{\sqrt{3} + 1}{2}$

$$\begin{aligned} \text{Ta có : } & \cos 12^\circ + \cos 18^\circ - 4 \cos 15^\circ (\cos 21^\circ \cos 24^\circ) \\ &= 2 \cos 15^\circ \cos 3^\circ - 2 \cos 15^\circ (\cos 45^\circ + \cos 3^\circ) \\ &= 2 \cos 15^\circ \cos 3^\circ - 2 \cos 15^\circ \cos 45^\circ - 2 \cos 15^\circ \cos 3^\circ \\ &= -2 \cos 15^\circ \cos 45^\circ \\ &= -(\cos 60^\circ + \cos 30^\circ) \\ &= -\frac{\sqrt{3} + 1}{2} \end{aligned}$$

Bài 17 : Tính $P = \sin^2 50^\circ + \sin^2 70^\circ - \cos 50^\circ \cos 70^\circ$

$$\begin{aligned} \text{Ta có : } P &= \frac{1}{2}(1 - \cos 100^\circ) + \frac{1}{2}(1 - \cos 140^\circ) - \frac{1}{2}(\cos 120^\circ + \cos 20^\circ) \\ P &= 1 - \frac{1}{2}(\cos 100^\circ + \cos 140^\circ) - \frac{1}{2}\left(-\frac{1}{2} + \cos 20^\circ\right) \\ P &= 1 - (\cos 120^\circ \cos 20^\circ) + \frac{1}{4} - \frac{1}{2} \cos 20^\circ \\ P &= \frac{5}{4} + \frac{1}{2} \cos 20^\circ - \frac{1}{2} \cos 20^\circ = \frac{5}{4} \end{aligned}$$

Bài 18 : Chứng minh : $\text{tg}30^\circ + \text{tg}40^\circ + \text{tg}50^\circ + \text{tg}60^\circ = \frac{8\sqrt{3}}{3} \cos 20^\circ$

$$\begin{aligned} \text{Áp dụng : } \text{tga} + \text{tgb} &= \frac{\sin(a+b)}{\cos a \cos b} \\ \text{Ta có : } & (\text{tg}50^\circ + \text{tg}40^\circ) + (\text{tg}30^\circ + \text{tg}60^\circ) \\ &= \frac{\sin 90^\circ}{\cos 50^\circ \cos 40^\circ} + \frac{\sin 90^\circ}{\cos 30^\circ \cos 60^\circ} \\ &= \frac{1}{\sin 40^\circ \cos 40^\circ} + \frac{1}{\frac{1}{2} \cos 30^\circ} \\ &= \frac{2}{\sin 80^\circ} + \frac{2}{\cos 30^\circ} \\ &= 2 \left(\frac{1}{\cos 10^\circ} + \frac{1}{\cos 30^\circ} \right) \\ &= 2 \left(\frac{\cos 30^\circ + \cos 10^\circ}{\cos 10^\circ \cos 30^\circ} \right) \\ &= 4 \frac{\cos 20^\circ \cos 10^\circ}{\cos 10^\circ \cos 30^\circ} \\ &= \frac{8\sqrt{3}}{3} \cos 20^\circ \end{aligned}$$

Bài 19 : Cho ΔABC , Chứng minh :

$$a/ \sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$b/ \cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$c/ \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$$

$$d/ \cos^2 A + \cos^2 B + \cos^2 C = -2 \cos A \cos B \cos C$$

$$e/ \operatorname{tg} A + \operatorname{tg} B + \operatorname{tg} C = \operatorname{tg} A \cdot \operatorname{tg} B \cdot \operatorname{tg} C$$

$$f/ \cot g A \cdot \cot g B + \cot g B \cdot \cot g C + \cot g C \cdot \cot g A = 1$$

$$g/ \cot g \frac{A}{2} + \cot g \frac{B}{2} + \cot g \frac{C}{2} = \cot g \frac{A}{2} \cdot \cot g \frac{B}{2} \cdot \cot g \frac{C}{2}$$

$$a/ \text{Ta có : } \sin A + \sin B + \sin C = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} + \sin(A+B)$$

$$= 2 \sin \frac{A+B}{2} \left(\cos \frac{A-B}{2} + \cos \frac{A+B}{2} \right)$$

$$= 4 \cos \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2} \quad \left(\text{do } \frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2} \right)$$

$$b/ \text{Ta có : } \cos A + \cos B + \cos C = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} - \cos(A+B)$$

$$= 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} - \left(2 \cos^2 \frac{A+B}{2} - 1 \right)$$

$$= 2 \cos \frac{A+B}{2} \left[\cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right] + 1$$

$$= -4 \cos \frac{A+B}{2} \sin \frac{A}{2} \sin \left(-\frac{B}{2} \right) + 1$$

$$= 4 \sin \frac{C}{2} \sin \frac{A}{2} \sin \frac{B}{2} + 1$$

$$c/ \sin 2A \sin 2B + \sin 2C = 2 \sin(A+B) \cos(A-B) + 2 \sin C \cos C$$

$$= 2 \sin C \cos(A-B) + 2 \sin C \cos C$$

$$= 2 \sin C [\cos(A-B) - \cos(A+B)]$$

$$= -4 \sin C \sin A \sin(-B)$$

$$= 4 \sin C \sin A \sin B$$

$$d/ \cos^2 A + \cos^2 B + \cos^2 C$$

$$= 1 + \frac{1}{2} (\cos 2A + \cos 2B) + \cos^2 C$$

$$= 1 + \cos(A+B) \cos(A-B) + \cos^2 C$$

$$= 1 - \cos C [\cos(A-B) - \cos C] \text{ do } (\cos(A+B) = -\cos C)$$

$$= 1 - \cos C [\cos(A-B) + \cos(A+B)]$$

$$= 1 - 2 \cos C \cdot \cos A \cdot \cos B$$

$$e/ \text{Do } a + b = \pi - C \text{ nên ta có}$$

$$\operatorname{tg}(A+B) = -\operatorname{tg} C$$

$$\Leftrightarrow \frac{\operatorname{tg}A + \operatorname{tg}B}{1 - \operatorname{tg}A\operatorname{tg}B} = -\operatorname{tg}C$$

$$\Leftrightarrow \operatorname{tg}A + \operatorname{tg}B = -\operatorname{tg}C + \operatorname{tg}A\operatorname{tg}B\operatorname{tg}C$$

$$\Leftrightarrow \operatorname{tg}A + \operatorname{tg}B + \operatorname{tg}C = \operatorname{tg}A\operatorname{tg}B\operatorname{tg}C$$

f/ Ta có : $\cotg(A+B) = -\cotgC$

$$\Leftrightarrow \frac{1 - \operatorname{tg}A\operatorname{tg}B}{\operatorname{tg}A + \operatorname{tg}B} = -\cotgC$$

$$\Leftrightarrow \frac{\cotgA \cotgB - 1}{\cotgB + \cotgA} = -\cotgC \quad (\text{nhân tử và mẫu cho } \cotgA.\cotgB)$$

$$\Leftrightarrow \cotgA \cotgB - 1 = -\cotgC \cotgB - \cotgA \cotgC$$

$$\Leftrightarrow \cotgA \cotgB + \cotgB \cotgC + \cotgA \cotgC = 1$$

$$g/ \text{Ta có : } \operatorname{tg} \frac{A+B}{2} = \cotg \frac{C}{2}$$

$$\Leftrightarrow \frac{\operatorname{tg} \frac{A}{2} + \operatorname{tg} \frac{B}{2}}{1 - \operatorname{tg} \frac{A}{2} \operatorname{tg} \frac{B}{2}} = \cotg \frac{C}{2}$$

$$\Leftrightarrow \frac{\cotg \frac{A}{2} + \cotg \frac{B}{2}}{\cotg \frac{A}{2} \cdot \cotg \frac{B}{2} - 1} = \cotg \frac{C}{2} \quad (\text{nhân tử và mẫu cho } \cotg \frac{A}{2} \cdot \cotg \frac{B}{2})$$

$$\Leftrightarrow \cotg \frac{A}{2} + \cotg \frac{B}{2} = \cotg \frac{A}{2} \cotg \frac{B}{2} \cotg \frac{C}{2} - \cotg \frac{C}{2}$$

$$\Leftrightarrow \cotg \frac{A}{2} + \cotg \frac{B}{2} + \cotg \frac{C}{2} = \cotg \frac{A}{2} \cdot \cotg \frac{B}{2} \cdot \cotg \frac{C}{2}$$

Bài 20 : Cho ΔABC . Chứng minh :

$$\cos 2A + \cos 2B + \cos 2C + 4\cos A \cos B \cos C + 1 = 0$$

Ta có : $(\cos 2A + \cos 2B) + (\cos 2C + 1)$

$$= 2 \cos(A+B) \cos(A-B) + 2\cos^2 C$$

$$= -2\cos C \cos(A-B) + 2\cos^2 C$$

$$= -2\cos C [\cos(A-B) + \cos(A+B)] = -4\cos A \cos B \cos C$$

Do đó : $\cos 2A + \cos 2B + \cos 2C + 1 + 4\cos A \cos B \cos C = 0$

Bài 21 : Cho ΔABC . Chứng minh :

$$\cos 3A + \cos 3B + \cos 3C = 1 - 4 \sin \frac{3A}{2} \sin \frac{3B}{2} \sin \frac{3C}{2}$$

Ta có : $(\cos 3A + \cos 3B) + \cos 3C$

$$= 2 \cos \frac{3}{2}(A+B) \cos \frac{3}{2}(A-B) + 1 - 2 \sin^2 \frac{3C}{2}$$

$$\text{Mà : } A+B = \pi - C \text{ nên } \frac{3}{2}(A+B) = \frac{3}{2}\pi - \frac{3C}{2}$$

$$\begin{aligned} \Rightarrow \cos \frac{3}{2}(A+B) &= \cos \left(\frac{3\pi}{2} - \frac{3C}{2} \right) \\ &= -\cos \left(\frac{\pi}{2} - \frac{3C}{2} \right) \\ &= -\sin \frac{3C}{2} \end{aligned}$$

Do đó : $\cos 3A + \cos 3B + \cos 3C$

$$\begin{aligned} &= -2 \sin \frac{3C}{2} \cos \frac{3(A-B)}{2} - 2 \sin^2 \frac{3C}{2} + 1 \\ &= -2 \sin \frac{3C}{2} \left[\cos \frac{3(A-B)}{2} + \sin \frac{3C}{2} \right] + 1 \\ &= -2 \sin \frac{3C}{2} \left[\cos \frac{3(A-B)}{2} - \cos \frac{3}{2}(A+B) \right] + 1 \\ &= 4 \sin \frac{3C}{2} \sin \frac{3A}{2} \sin \left(\frac{-3B}{2} \right) + 1 \\ &= -4 \sin \frac{3C}{2} \sin \frac{3A}{2} \sin \frac{3B}{2} + 1 \end{aligned}$$

Bài 22 : A, B, C là ba góc của một tam giác. Chứng minh :

$$\frac{\sin A + \sin B - \sin C}{\cos A + \cos B - \cos C + 1} = \operatorname{tg} \frac{A}{2} \operatorname{tg} \frac{B}{2} \cot g \frac{C}{2}$$

Ta có :

$$\begin{aligned} \frac{\sin A + \sin B - \sin C}{\cos A + \cos B - \cos C + 1} &= \frac{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} - 2 \sin \frac{C}{2} \cos \frac{C}{2}}{2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} + 2 \sin^2 \frac{C}{2}} \\ &= \frac{2 \cos \frac{C}{2} \left[\cos \frac{A-B}{2} - \sin \frac{C}{2} \right]}{2 \sin \frac{C}{2} \left[\cos \frac{A-B}{2} + \sin \frac{C}{2} \right]} = \cot g \frac{C}{2} \cdot \frac{\cos \frac{A-B}{2} - \cos \frac{A+B}{2}}{\cos \frac{A-B}{2} + \cos \frac{A+B}{2}} \\ &= \cot g \frac{C}{2} \cdot \frac{-2 \sin \frac{A}{2} \cdot \sin \left(\frac{-B}{2} \right)}{2 \cos \frac{A}{2} \cdot \cos \frac{B}{2}} \\ &= \cot g \frac{C}{2} \cdot \operatorname{tg} \frac{A}{2} \cdot \operatorname{tg} \frac{B}{2} \end{aligned}$$

Bài 23 : Cho ΔABC . Chứng minh :

$$\begin{aligned} &\sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} + \sin \frac{B}{2} \cos \frac{C}{2} \cos \frac{A}{2} + \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2} \\ &= \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + \operatorname{tg} \frac{A}{2} \operatorname{tg} \frac{B}{2} + \operatorname{tg} \frac{B}{2} \operatorname{tg} \frac{C}{2} + \operatorname{tg} \frac{A}{2} \operatorname{tg} \frac{C}{2} (*) \end{aligned}$$

Ta có : $\frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}$ vậy $\operatorname{tg}\left(\frac{A}{2} + \frac{B}{2}\right) = \operatorname{cotg} \frac{C}{2}$

$$\Leftrightarrow \frac{\operatorname{tg} \frac{A}{2} + \operatorname{tg} \frac{B}{2}}{1 - \operatorname{tg} \frac{A}{2} \operatorname{tg} \frac{B}{2}} = \frac{1}{\operatorname{tg} \frac{C}{2}}$$

$$\Leftrightarrow \left[\operatorname{tg} \frac{A}{2} + \operatorname{tg} \frac{B}{2} \right] \operatorname{tg} \frac{C}{2} = 1 - \operatorname{tg} \frac{A}{2} \operatorname{tg} \frac{B}{2}$$

$$\Leftrightarrow \operatorname{tg} \frac{A}{2} \operatorname{tg} \frac{C}{2} + \operatorname{tg} \frac{B}{2} \operatorname{tg} \frac{C}{2} + \operatorname{tg} \frac{A}{2} \operatorname{tg} \frac{B}{2} = 1 \quad (1)$$

Do đó : (*) $\Leftrightarrow \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} + \sin \frac{B}{2} \cos \frac{C}{2} \cos \frac{A}{2} + \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2}$
 $= \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + 1$ (do (1))

$$\Leftrightarrow \sin \frac{A}{2} \left[\cos \frac{B}{2} \cos \frac{C}{2} - \sin \frac{B}{2} \sin \frac{C}{2} \right] + \cos \frac{A}{2} \left[\sin \frac{B}{2} \cos \frac{C}{2} + \sin \frac{C}{2} \cos \frac{B}{2} \right] = 1$$

$$\Leftrightarrow \sin \frac{A}{2} \cos \frac{B+C}{2} + \cos \frac{A}{2} \sin \frac{B+C}{2} = 1$$

$$\Leftrightarrow \sin \frac{A+B+C}{2} = 1 \Leftrightarrow \sin \frac{\pi}{2} = 1 \quad (\text{hiển nhiên đúng})$$

Bài 24: Chứng minh : $\operatorname{tg} \frac{A}{2} + \operatorname{tg} \frac{B}{2} + \operatorname{tg} \frac{C}{2} = \frac{3 + \cos A + \cos B + \cos C}{\sin A + \sin B + \sin C}$ (*)

Ta có :

$$\begin{aligned} \cos A + \cos B + \cos C + 3 &= 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} + \left[1 - 2 \sin^2 \frac{C}{2} \right] + 3 \\ &= 2 \sin \frac{C}{2} \cos \frac{A-B}{2} + 4 - 2 \sin^2 \frac{C}{2} \\ &= 2 \sin \frac{C}{2} \left[\cos \frac{A-B}{2} - \sin \frac{C}{2} \right] + 4 \\ &= 2 \sin \frac{C}{2} \left[\cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right] + 4 \\ &= 4 \sin \frac{C}{2} \sin \frac{A}{2} \cdot \sin \frac{B}{2} + 4 \quad (1) \end{aligned}$$

$$\begin{aligned} \sin A + \sin B + \sin C &= 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} + \sin C \\ &= 2 \cos \frac{C}{2} \cos \frac{A-B}{2} + 2 \sin \frac{C}{2} \cos \frac{C}{2} \\ &= 2 \cos \frac{C}{2} \left[\cos \frac{A-B}{2} + \cos \frac{A+B}{2} \right] \\ &= 4 \cos \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2} \quad (2) \end{aligned}$$

Từ (1) và (2) ta có :

$$\begin{aligned}
(*) &\Leftrightarrow \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} + \frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} = \frac{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + 1}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} \\
&\Leftrightarrow \sin \frac{A}{2} \left[\cos \frac{B}{2} \cos \frac{C}{2} \right] + \sin \frac{B}{2} \left[\cos \frac{A}{2} \cos \frac{C}{2} \right] + \sin \frac{C}{2} \left[\cos \frac{A}{2} \cos \frac{B}{2} \right] \\
&\hspace{20em} = \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + 1 \\
&\Leftrightarrow \sin \frac{A}{2} \left[\cos \frac{B}{2} \cos \frac{C}{2} - \sin \frac{B}{2} \sin \frac{C}{2} \right] + \cos \frac{A}{2} \left[\sin \frac{B}{2} \cos \frac{C}{2} + \sin \frac{C}{2} \cos \frac{B}{2} \right] = 1 \\
&\Leftrightarrow \sin \frac{A}{2} \cdot \cos \frac{B+C}{2} + \cos \frac{A}{2} \sin \frac{B+C}{2} = 1 \\
&\Leftrightarrow \sin \left[\frac{A+B+C}{2} \right] = 1 \\
&\Leftrightarrow \sin \frac{\pi}{2} = 1 \text{ (hiển nhiên đúng)}
\end{aligned}$$

Bài 25 : Cho ΔABC . Chứng minh: $\frac{\sin \frac{A}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} + \frac{\sin \frac{B}{2}}{\cos \frac{C}{2} \cos \frac{A}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} = 2$

Cách 1 :

Ta có :

$$\begin{aligned}
&\frac{\sin \frac{A}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} + \frac{\sin \frac{B}{2}}{\cos \frac{C}{2} \cos \frac{A}{2}} = \frac{\sin \frac{A}{2} \cos \frac{A}{2} + \sin \frac{B}{2} \cos \frac{B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} \\
&= \frac{1}{2} \frac{\sin A + \sin B}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{\sin \frac{A+B}{2} \cos \frac{A-B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} \\
&= \frac{\cos \frac{C}{2} \cdot \cos \frac{A-B}{2}}{\cos \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2}} = \frac{\cos \left(\frac{A-B}{2} \right)}{\cos \frac{A}{2} \cos \frac{B}{2}} \\
\text{Do đó : Vế trái} &= \frac{\cos \left(\frac{A-B}{2} \right)}{\cos \frac{A}{2} \cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} = \frac{\cos \frac{A-B}{2} + \cos \frac{A+B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} \\
&= \frac{2 \cos \frac{A}{2} \cos \frac{B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} = 2
\end{aligned}$$

Cách 2 :

$$\begin{aligned} \text{Ta có vế trái} &= \frac{\cos \frac{B+C}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} + \frac{\cos \frac{A+C}{2}}{\cos \frac{C}{2} \cos \frac{A}{2}} + \frac{\cos \frac{A+B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} \\ &= \frac{\cos \frac{B}{2} \cos \frac{C}{2} - \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} + \frac{\cos \frac{A}{2} \cos \frac{C}{2} - \sin \frac{A}{2} \sin \frac{C}{2}}{\cos \frac{C}{2} \cos \frac{A}{2}} \\ &\quad + \frac{\cos \frac{A}{2} \cos \frac{B}{2} - \sin \frac{A}{2} \sin \frac{B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} \end{aligned}$$

$$= 3 - \left[\operatorname{tg} \frac{B}{2} \operatorname{tg} \frac{C}{2} + \operatorname{tg} \frac{A}{2} \operatorname{tg} \frac{C}{2} + \operatorname{tg} \frac{A}{2} \operatorname{tg} \frac{B}{2} \right]$$

Mà : $\operatorname{tg} \frac{A}{2} \operatorname{tg} \frac{B}{2} + \operatorname{tg} \frac{B}{2} \operatorname{tg} \frac{C}{2} + \operatorname{tg} \frac{A}{2} \operatorname{tg} \frac{C}{2} = 1$

(đã chứng minh tại bài 10)

Do đó : $\text{Vế trái} = 3 - 1 = 2$

Bài 26: Cho ΔABC . Có $\cot g \frac{A}{2}, \cot g \frac{B}{2}, \cot g \frac{C}{2}$ theo tử tự tạo cấp số cộng.

Chứng minh $\cot g \frac{A}{2} \cdot \cot g \frac{C}{2} = 3$

Ta có : $\cot g \frac{A}{2}, \cot g \frac{B}{2}, \cot g \frac{C}{2}$ là cấp số cộng

$$\Leftrightarrow \cot g \frac{A}{2} + \cot g \frac{C}{2} = 2 \cot g \frac{B}{2}$$

$$\Leftrightarrow \frac{\sin \frac{A+C}{2}}{\sin \frac{A}{2} \sin \frac{C}{2}} = \frac{2 \cos \frac{B}{2}}{\sin \frac{B}{2}}$$

$$\Leftrightarrow \frac{\cos \frac{B}{2}}{\sin \frac{A}{2} \sin \frac{C}{2}} = \frac{2 \cos \frac{B}{2}}{\sin \frac{B}{2}}$$

$$\Leftrightarrow \frac{1}{\sin \frac{A}{2} \sin \frac{C}{2}} = \frac{2}{\cos \frac{A+C}{2}} \quad (\text{do } 0 < B < \pi \text{ nên } \cos \frac{B}{2} > 0)$$

$$\Leftrightarrow \frac{\cos \frac{A}{2} \cos \frac{C}{2} - \sin \frac{A}{2} \sin \frac{C}{2}}{\sin \frac{A}{2} \cdot \sin \frac{C}{2}} = 2 \Leftrightarrow \cot g \frac{A}{2} \cot g \frac{C}{2} = 3$$

Bài 27: Cho ΔABC . Chứng minh :

$$\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} = \frac{1}{2} \left[\operatorname{tg} \frac{A}{2} + \operatorname{tg} \frac{B}{2} + \operatorname{tg} \frac{C}{2} + \operatorname{cotg} \frac{A}{2} + \operatorname{cotg} \frac{B}{2} + \operatorname{cotg} \frac{C}{2} \right]$$

$$\text{Ta có : } \operatorname{cotg} \frac{A}{2} + \operatorname{cotg} \frac{B}{2} + \operatorname{cotg} \frac{C}{2} = \operatorname{cotg} \frac{A}{2} \cdot \operatorname{cotg} \frac{B}{2} \cdot \operatorname{cotg} \frac{C}{2}$$

(Xem chứng minh bài 19g)

$$\text{Mặt khác : } \operatorname{tg} \alpha + \operatorname{cotg} \alpha = \frac{\sin \alpha}{\cos \alpha} + \frac{\cos \alpha}{\sin \alpha} = \frac{2}{\sin 2\alpha}$$

$$\begin{aligned} \text{Do đó : } & \frac{1}{2} \left[\operatorname{tg} \frac{A}{2} + \operatorname{tg} \frac{B}{2} + \operatorname{tg} \frac{C}{2} + \operatorname{cotg} \frac{A}{2} + \operatorname{cotg} \frac{B}{2} + \operatorname{cotg} \frac{C}{2} \right] \\ &= \frac{1}{2} \left[\operatorname{tg} \frac{A}{2} + \operatorname{tg} \frac{B}{2} + \operatorname{tg} \frac{C}{2} \right] + \frac{1}{2} \left[\operatorname{cotg} \frac{A}{2} + \operatorname{cotg} \frac{B}{2} + \operatorname{cotg} \frac{C}{2} \right] \\ &= \frac{1}{2} \left[\operatorname{tg} \frac{A}{2} + \operatorname{cotg} \frac{A}{2} \right] + \frac{1}{2} \left[\operatorname{tg} \frac{B}{2} + \operatorname{cotg} \frac{B}{2} \right] + \frac{1}{2} \left[\operatorname{tg} \frac{C}{2} + \operatorname{cotg} \frac{C}{2} \right] \\ &= \frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} \end{aligned}$$

BÀI TẬP

1. Chứng minh :

$$a/ \cos \frac{\pi}{5} - \cos \frac{2\pi}{5} = \frac{1}{2}$$

$$b/ \frac{\cos 15^\circ + \sin 15^\circ}{\cos 15^\circ - \sin 15^\circ} = \sqrt{3}$$

$$c/ \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} = -\frac{1}{2}$$

$$d/ \sin^3 2x \sin 6x + \cos^3 2x \cdot \cos 6x = \cos^3 4x$$

$$e/ \operatorname{tg} 20^\circ \cdot \operatorname{tg} 40^\circ \cdot \operatorname{tg} 60^\circ \cdot \operatorname{tg} 80^\circ = 3$$

$$f/ \operatorname{tg} \frac{\pi}{6} + \operatorname{tg} \frac{2\pi}{9} + \operatorname{tg} \frac{5\pi}{18} + \operatorname{tg} \frac{\pi}{3} = \frac{8\sqrt{3}}{3} \cos \frac{\pi}{9}$$

$$g/ \cos \frac{\pi}{15} \cdot \cos \frac{2\pi}{15} \cdot \cos \frac{3\pi}{15} \cdot \cos \frac{4\pi}{15} \cdot \cos \frac{5\pi}{15} \cdot \cos \frac{6\pi}{15} \cdot \cos \frac{7\pi}{15} = \frac{1}{2^7}$$

$$h/ \operatorname{tg} x \cdot \operatorname{tg} \left[\frac{\pi}{3} - x \right] \cdot \operatorname{tg} \left[\frac{\pi}{3} + x \right] = \operatorname{tg} 3x$$

$$k/ \operatorname{tg} 20^\circ + \operatorname{tg} 40^\circ + \sqrt{3} \operatorname{tg} 20^\circ \cdot \operatorname{tg} 40^\circ = \sqrt{3}$$

$$e/ \sin 20^\circ \cdot \sin 40^\circ \cdot \sin 80^\circ = \frac{\sqrt{3}}{8}$$

$$m/ \operatorname{tg} 5^\circ \cdot \operatorname{tg} 55^\circ \cdot \operatorname{tg} 65^\circ \cdot \operatorname{tg} 75^\circ = 1$$

$$2. \text{ Chứng minh rằng nếu } \begin{cases} \sin x = 2 \sin(x + y) \\ x + y \neq (2k + 1) \frac{\pi}{2} \quad (k \in \mathbb{Z}) \end{cases}$$

$$\text{thì } \operatorname{tg}(x + y) = \frac{\sin y}{\cos y - 2}$$

3. Cho ΔABC có 3 góc đều nhọn và $A \geq B \geq C$

a/ Chứng minh : $\operatorname{tg}A + \operatorname{tg}B + \operatorname{tg}C = \operatorname{tg}A \cdot \operatorname{tg}B \cdot \operatorname{tg}C$

b/ Đặt $\operatorname{tg}A \cdot \operatorname{tg}B = p$; $\operatorname{tg}A \cdot \operatorname{tg}C = q$

Chứng minh $(p-1)(q-1) \geq 4$

4. Chứng minh các biểu thức không phụ thuộc x :

a/ $A = \sin^4 x (1 + \sin^2 x) + \cos^4 x (1 + \cos^2 x) + 5 \sin^2 x \cos^2 x + 1$

b/ $B = 3(\sin^8 x - \cos^8 x) + 4(\cos^6 x - 2\sin^6 x) + 6\sin^4 x$

c/ $C = \cos^2(x - a) + \sin^2(x - b) - 2\cos(x - a)\sin(x - b)\sin(a - b)$

5. Cho ΔABC , chứng minh :

a/ $\cot gB + \frac{\cos C}{\sin B \cos A} = \cot gC + \frac{\cos B}{\sin C \cos A}$

b/ $\sin^3 A + \sin^3 B + \sin^3 C = 3\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} + \cos \frac{3A}{2} \cos \frac{3B}{2} \cos \frac{3C}{2}$

c/ $\sin A + \sin B + \sin C = \cos \frac{A}{2} \cdot \cos \frac{B-C}{2} + \cos \frac{B}{2} \cdot \cos \frac{A-C}{2} + \cos \frac{C}{2} \cdot \cos \frac{A-B}{2}$

d/ $\cot gA \cot gB + \cot gB \cot gC + \cot gC \cot gA = 1$

e/ $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$

f/ $\sin 3A \sin(B - C) + \sin 3B \sin(C - A) + \sin 3C \sin(A - B) = 0$

6. Tìm giá trị nhỏ nhất của :

a/ $y = \frac{1}{\sin x} + \frac{1}{\cos x}$ với $0 < x < \frac{\pi}{2}$

b/ $y = 4x + \frac{9\pi}{x} + \sin x$ với $0 < x < \infty$

c/ $y = 2\sin^2 x + 4\sin x \cos x + \sqrt{5}$

7. Tìm giá trị lớn nhất của :

a/ $y = \sin x \sqrt{\cos x} + \cos x \sqrt{\sin x}$

b/ $y = \sin x + 3\sin 2x$

c/ $y = \cos x + \sqrt{2 - \cos^2 x}$